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Highest-weight representations of the sl($n+1$, $\mathbb{C}$) algebras: maximal representations

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Abstract. Representations of the sl($n+1$, $\mathbb{C}$) Lie algebras are constructed with the help of canonical (boson) realisations of these algebras. For every weight Λ on the standard Cartan subalgebra of sl($n+1$, $\mathbb{C}$) we obtain a representation $\rho_\Lambda^{(n+1)}$ (called the maximal representation) which contains an irreducible subrepresentation with Λ as the highest weight. It is shown that for a major part of the weights Λ (specified by theorem 4.3) the representations $\rho_\Lambda^{(n+1)}$ themselves are irreducible. The standard construction of the highest-weight representations of semi-simple Lie algebras is based on the so-called elementary representations; comparing with them, our maximal representations are given in the explicit form.

1. Introduction

1.1

There are essentially two reasons which make the highest-weight representations of semi-simple Lie algebras interesting. The first of them concerns their applications in quantum mechanics and elementary particle physics (reviewed e.g. by Gruber and Klimyk (1975)). On the other hand, mathematically they are a generalisation of the finite-dimensional irreducible representations conserving some of their properties.

1.2

The finite-dimensional irreducible representations with a highest weight Λ of a complex semi-simple Lie algebra L are characterised by the condition that $\Lambda_i \equiv 2(\Lambda, \omega_i)/(\omega_i, \omega_i)$, $i = 1, 2, \ldots, n$, are equal to non-negative integers (Zhelobenko 1970); here ω_i and n are positive simple roots and the rank of L, respectively. Properties of these representations are well known (Zhelobenko 1970, Naimark 1976, Dixmier 1974). The representations with Λ_i arbitrary integers may be infinite-dimensional, but remain integrable; this case was studied by Harish-Chandra (1955).

The representations mentioned form, of course, only a small part among all the highest-weight representations of a given L. Many results concerning the general case (with no restrictions on Λ) can be deduced from the theory of Verma modules (Dixmier 1974); an extensive treatment of this problem was carried out by Gruber and Klimyk (1975). In their paper the so-called elementary representations were introduced and

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studied (cf § 2.6; essentially the same construction was used also by other authors for investigation of the highest-weight representations). The elementary representations are *ex definitio* representations with a highest weight; in general, they need not be irreducible; however, they are irreducible for a 'great' subset in the set of all weights Λ.

Since there is a one-to-one correspondence between the weights Λ and the irreducible highest-weight representations of L (cf theorem 2.4(b)), it might seem that no other highest-weight representations are needed, at least for those Λ for which the elementary representations are irreducible. However, the representation spaces of the elementary representations are certain factor spaces (cf § 2.6). It makes their use extremely difficult even in the case of the lowest-dimensional algebras, and represents a great practical disadvantage. This is why we suppose a search for other irreducible highest-weight representations to be meaningful.

1.3

In this paper we shall give another set of irreducible highest-weight represetations of $\mathrm{sl}(n+1, \mathbb{C})$. A major part of them will be obtained in the explicit form in which matrix elements of generators can be easily calculated. In a subsequent paper (Burdík *et al* (1981) we shall illustrate, using the example of $\mathrm{sl}(3, \mathbb{C}) \sim A_2$, that such explicit representations are given for all the weights Λ to which the irreducible elementary representations correspond. Moreover, we shall demonstrate that our method makes it possible to construct irreducible highest-weight representations also for some of the weights Λ such that the corresponding elementary representations are reducible.

1.4

The construction presented in the following sections is based on canonical (or boson) realisations of $\mathrm{sl}(n+1, \mathbb{C})$. Various applications of the boson operator technique to group representation theory are well known: see e.g. Baird and Biedenharn (1963), Kihlberg (1965), Moshinsky (1962) or Barut and Raczka (1977, ch X and references therein). Our method starts from the deeper and more systematic study of canonical realisations (Havlíček and Exner 1975a,b, 1978, Havlíček and Lassner 1975, 1976a,b,c, 1977); for a particular survey of the subject see Exner *et al* (1976)). In these papers new wide families of realisations were derived for all complex classical Lie algebras as well as for the majority of their real forms. Compared with most of the standardly used ones, they have the Casimir operators realised by multiples of unity and enough parameters to make values of all the generating Casimirs independent. Moreover, realisations of the real forms are skew-symmetric under some standard involution.

In this paper we treat the algebras $\mathrm{sl}(n+1, \mathbb{C}) \sim A_n$ because their realisations are the simplest among the abovementioned ones. Let us notice that the particular case $n = 3$ is of some interest in connection with the recent attempt to combine gravity with strong interactions (Ne'eman and Šijački 1979). We believe that our realisations can be used to construct highest-weight representations for other semi-simple Lie algebras as well; some positive indications for the B_n and D_n algebras have been obtained already.

1.5

The paper is organised as follows. All the necessary prerequisites are listed in § 2. The next two sections contain the main results. In § 3 the maximal representations $\rho_{\Lambda}^{(n+1)}$ of sl$(n+1, \mathbb{C})$ are constructed; it is shown further that each $\rho_{\Lambda}^{(n+1)}$ contains an irreducible subrepresentation $\tilde{Q}_{\Lambda}^{(n+1)}$ with the highest weight Λ. In § 4 conditions are given under which the maximal representations themselves are irreducible. In the last section the results are discussed, and in particular a detailed comparison with the elementary representations is made. In the subsequent paper mentioned above, the results will be illustrated using the simplest examples of the A_1 and A_2 algebras.

2. Preliminaries

2.1

The algebra gl$(n+1, \mathbb{C})$ is the $(n+1)^2$-dimensional complex Lie algebra with the standard basis $\{e_{ij}: i, j = 1, 2, \ldots, n+1\}$, the elements of which obey

$$[e_{ij}, e_{kl}] = \delta_{kj} e_{il} - \delta_{il} e_{kj}. \tag{1}$$

This algebra is a direct sum of its one-dimensional centrum (generated by the element $e = \Sigma_{i=1}^{n+1} e_{ii}$) and the simple subalgebra sl$(n+1, \mathbb{C}) \sim A_n$ whose generators are e_{ij}, $i \neq j$ and $a_i = e_{ii} - (1/n)e$, $i = 1, 2, \ldots, n$.

2.2

The standard Cartan subalgebra H in L = sl$(n+1, \mathbb{C})$ is generated by the 'diagonal' elements a_i; its dimension, i.e. the rank of L, equals n. We choose the following Cartan–Weyl basis:

$$h_i = a_{i+1} - a_i = e_{i+1,i+1} - e_{ii}, \qquad i = 1, 2, \ldots, n, \tag{2a}$$

$$e_i = e_{i+1,i}, \qquad i = 1, 2, \ldots, n, \tag{2b}$$

$$f_i = e_{-i} = e_{i,i+1}, \qquad i = 1, 2, \ldots, n, \tag{2c}$$

$$e_{ij}, \qquad i > j+1, \tag{2d}$$

$$e_{ij}, \qquad i < j-1. \tag{2e}$$

The relations (1) imply that ($2b$–$2e$) are the root vectors corresponding to the roots α_{ij}: $\alpha_{ij}(\Sigma_{k=1}^{n} \lambda_k h_k) = \lambda_i - \lambda_j$. Among these roots $\omega_i \equiv \alpha_{i+1,i}$ and $\omega_{-i} \equiv \alpha_{i,i+1}$ are simple; further, α_{ij}, $i > j$, are positive. Following Zhelobenko (1970), we call the elements ($2a$–$2c$) canonical generators of L. They satisfy the relations

$$[h_i, h_j] = 0, \tag{3a}$$

$$[e_i, f_j] = \delta_{ij} h_i, \tag{3b}$$

$$[h_i, e_j] = c_{ij} e_j, \qquad [h_i, f_j] = -c_{ij} f_j, \tag{3c}$$

where c_{ij} are the Cartan numbers, $c_{ij} = 2, -1, 0$ for $i = j$, $|i-j| = 1$, $|i-j| > 1$, respectively. Notice that the Cartan–Weyl basis (2) differs from the standard one (Bourbaki 1968); they are connected by the automorphism generated by $e_{\pm i} \mapsto e_{\mp i}$, $h_i \mapsto h_{-i}$. We choose the basis (2) because it is suitable for our construction.

2.3

The universal enveloping algebra of L will be denoted conventionally by UL. Let ρ be a representation of L on a vector space V; by the same symbol we denote also the natural extension of ρ to UL. A representation $\rho : \mathrm{L} \to \mathscr{L}(V)$ is called the representation with a highest weight $\Lambda = (\Lambda_1, \ldots, \Lambda_n)$ if there exists a vector $x_0 \in V$ (called the highest-weight vector) such that the following three conditions are fulfilled:

(i) the linear form Λ on H, $\Lambda(h_i) = \Lambda_i$, is a weight of ρ, it holds that $\rho(h)x_0 = \Lambda(h)x_0$ for all $h \in \mathrm{H}$ or equivalently $\rho(h_i)x_0 = \Lambda_i x_0$, $i = 1, 2, \ldots, n$;

(ii) $\rho(e_i)x_0 = 0$, $i = 1, 2, \ldots, n$;

(iii) the vector x_0 is cyclic for ρ, i.e. $\rho(\mathrm{UL})x_0 \equiv \{\rho(a)x_0 : a \in \mathrm{UL}\} = V$.

Since a system of canonical generators exists in any semi-simple Lie algebra, this definition applies not only to $\mathrm{L} = \mathrm{sl}(n+1, \mathbb{C})$ but to the other semi-simple algebras as well, The lowest-weight representations are defined in the same way, the only change consisting of replacement of $\rho(e_i)$ by $\rho(f_i)$ in (ii). Some important properties of the highest-weight representations are summarised in the following assertions (cf e.g. Gruber and Klimyk (1975) for references to proofs).

2.4. Theorem.

Let L be a complex semi-simple Lie algebra and $\rho : \mathrm{L} \to \mathscr{L}(V)$ its representation with a highest weight Λ.

(a) The space V decomposes into a direct sum of finite-dimensional weight subspaces $V_M \equiv \{x \in V : \rho(h)x = M(h)x, \forall h \in \mathrm{H}\}$, the subspace V_Λ being one-dimensional. Every weight M of ρ is of the form $M = \Lambda - \Sigma_{i=1}^{n} k_i \omega_i$, where ω_i are the positive simple roots of L and k_i are non-negative integers.

(b) For each linear form Λ on the Cartan subalgebra H of L there exists, up to equivalence, one and only one irreducible representation ρ of L with Λ as the highest weight.

2.5. Theorem.

Let the assumptions of the previous theorem be valid. The representation ρ is finite-dimensional if and only if $\Lambda_i = \Lambda(h_i)$, $i = 1, 2, \ldots, n$, are non-negative integers.

2.6

Now we shall define the elementary representations of L. The algebra L decomposes into the direct sum $\mathrm{L} = \mathrm{L}_+ + \mathrm{H} + \mathrm{L}_-$ where L_- is the subalgebra generated by the elements f_i (cf (2*c*); notice that each of the elements (2*e*) can be obtained from $f_1, \ldots, f_n$ by Lie products). The universal enveloping algebra UL_- of L_- serves as a representation space. It can be identified with the free algebra of monomials

$$1, f_{i_1} f_{i_2} \ldots f_{i_m}, \qquad i_k = 1, 2, \ldots, n, \qquad m = 1, 2, \ldots,$$

factorised by the ideal generated by the following elements:

$$[\ldots[[f_{i_1}, f_{i_2}], f_{i_3}], \ldots f_{i_m}], \qquad m = 2, 3, \ldots,$$

for those $(i_1, i_2, \ldots, i_m)$ for which the sum of positive simple roots $\Sigma_{k=1}^{m} \omega_{i_k}$ is a root.

The elementary representation d_Λ corresponding to a linear form Λ on H is defined by the following relations†

$$d_\Lambda(h)1 = \Lambda(h)1, \qquad d_\Lambda(f_i)1 = f_i, \qquad d_\Lambda(e_i)1 = 0, \tag{4a}$$

$$d_\Lambda(h)f_{i_1}f_{i_2}\ldots f_{i_m} = (\Lambda - \omega_{i_1} - \omega_{i_2} - \ldots - \omega_{i_m})(h)f_{i_1}f_{i_2}\ldots f_{i_m}, \tag{4b}$$

$$d_\Lambda(f_i)f_{i_1}f_{i_2}\ldots f_{i_m} = f_if_{i_1}f_{i_2}\ldots f_{i_m}, \tag{4c}$$

$$d_\Lambda(e_i)f_{i_1}f_{i_2}\ldots f_{i_m} = f_{i_1}(d_\Lambda(e_i)f_{i_2}\ldots f_{i_m}) + \delta_{ii_1}(\Lambda - \omega_{i_2} - \ldots - \omega_{i_m})(h_i)f_{i_2}\ldots f_{i_m}; \tag{4d}$$

here ω_i are again the positive simple roots of L. The representation d_Λ is clearly a representation with the highest weight Λ; in general it is reducible but not completely reducible. Necessary and sufficient conditions for irreducibility of d_Λ can be found which employ the action of the Weyl group W of L on the highest weight Λ (cf theorems 5, 6 of Gruber and Klimyk (1975)).

2.7

The last introductory item concerns the canonical realisations which are the basic tool of our construction. The (complex) Weyl algebra W_{2N} is the associative algebra with unity 1 generated by the elements $q_i, p_j, i, j = 1, 2, \ldots, N$, which obey

$$[p_i, p_j] = [q_i, q_j] = 0, \qquad [p_i, q_j] = \delta_{ij}1;$$

it is often called a *boson algebra* in physical literature. A *canonical (or boson) realisation* of a Lie algebra L is a homomorphism $L \to W_{2N}$; it extends naturally to the homomorphism $UL \to W_{2N}$. In the following we shall deal with simple algebras; in this case any non-trivial realisation is injective. For further notions and properties concerning canonical realisations we refer to the papers by two of us (together with W Lassner) quoted in § 1.4.

We shall use the canonical realisations of gl$(n+1, \mathbb{C})$ constructed by Havlíček and Lassner (1975). They are obtained recursively with the help of n canonical pairs, one complex parameter and a realisation of gl$(n, \mathbb{C})$ (however, this is not a realisation of the corresponding gl$(n, \mathbb{C})$ subalgebra in gl$(n+1, \mathbb{C})$—cf (5) below—and therefore there is no direct analogy here to what are called canonical bases by Gruber and Klimyk (1979)). The latter can be chosen in different ways: canonical realisation of the same type, matrix representation or trivial representation; in the first case the same possibilities appear after the next step in the choice of a realisation of gl$(n-1, \mathbb{C})$ etc. In what follows we employ mostly the first possibility when the reduction is performed to the end with canonical realisations of the same type (the possibility of using matrix representations of some subalgebra will be employed in the subsequent paper mentioned in § 1.3). The realisation of the generators e_{ij} of gl$(k+1, \mathbb{C})$ will be denoted by $\tau^{(k+1)}(e_{ij})$. It is convenient to enumerate the canonical pairs in these realisations by two indices: $q_i^{k+1}, p_j^{k+1}, i, j = 1, 2, \cdots, k, \; k = 2, 3, \cdots$; then the following assertion is valid.

2.8. Proposition.

For any complex numbers $\alpha_0, \alpha_1, \ldots, \alpha_n$ there exists a realisation of gl$(n+1, \mathbb{C})$ in

† The difference in sign in (4d) compared with Gruber and Klimyk (1975) is due to a slightly different choice of the canonical generators.

W_{2N}, $N=\frac{1}{2}(n+1)(n+2)$; it is given recursively by the formulae

$$\tau^{(n+1)}(e_{ij})=q_i^{n+1}p_j^{n+1}+\tau^{(n)}(e_{ij})+\tfrac{1}{2}\delta_{ij}1, \tag{5a}$$

$$\tau^{(n+1)}(e_{n+1,i})=-p_i^{n+1}, \tag{5b}$$

$$\tau^{(n+1)}(e_{i,n+1})=q_i^{n+1}\left(\sum_{j=1}^{n} q_j^{n+1}p_j^{n+1}+\frac{n+1}{2}-\mathrm{i}\alpha_n\right)+\sum_{j=1}^{n} q_j^{n+1}\tau^{(n)}(e_{ij}), \tag{5c}$$

$$\tau^{(n+1)}(e_{n+1,n+1})=-\sum_{j=1}^{n} q_j^{n+1}p_j^{n+1}-\left(\frac{n}{2}-\mathrm{i}\alpha_n\right)1, \tag{5d}$$

$i,j=1,2,\ldots,n$, where $\tau^{(1)}(e_{11})=\mathrm{i}\alpha_0$.

3. Maximal representations of sl($n+1$, $\mathbb{C}$)

3.1

Let B_{n+1} denote the set of all symbols (triangular matrices)

$$\begin{vmatrix} m_{n1} & m_{n2} & & \cdots & & m_{nn} \\ & & \cdots & & & \\ m_{k1} & m_{k2} & \cdots & m_{kk} & & \\ & \cdots & & & & \\ m_{21} & m_{22} & & & & \\ m_{11} & & & & & \end{vmatrix}, \qquad m_{kj}\in N_0, \tag{6}$$

where N_0 is the set of all non-negative integers. These symbols will denote the basis vectors and the complex linear envelope $V_{n+1}=\mathbb{C}\{B_{n+1}\}$ will serve as the representation space. The vector

$$x_0^{n+1}=\begin{vmatrix} 0 & \cdots & 0 \\ & \cdots & \\ 0 & & \end{vmatrix}$$

is called the vacuum vector. We define the creation and annihilation operators $\bar{a}_j^{k+1}$, a_j^{k+1}, $j=1,\ldots,k$, $k=1,\ldots,n$, on V_{n+1} in the standard way:

$$\bar{a}_j^{k+1}\begin{vmatrix} m_{n1} & & \cdots & & m_{nn} \\ & & \cdots & & \\ m_{k1} & \cdots & m_{kj} & \cdots & m_{kk} \\ & \cdots & & & \\ m_{11} & & & & \end{vmatrix} = (m_{kj}+1)^{1/2}\begin{vmatrix} m_{n1} & & \cdots & & m_{nn} \\ & & \cdots & & \\ m_{k1} & \cdots & m_{kj}+1 & \cdots & m_{kk} \\ & \cdots & & & \\ m_{11} & & & & \end{vmatrix}, \tag{7a}$$

$$a_j^{k+1}\begin{vmatrix} m_{n1} & & \cdots & & m_{nn} \\ & & \cdots & & \\ m_{k1} & \cdots & m_{kj} & \cdots & m_{kk} \\ & \cdots & & & \\ m_{11} & & & & \end{vmatrix} = (m_{kj})^{1/2}\begin{vmatrix} m_{n1} & & \cdots & & m_{nn} \\ & & \cdots & & \\ m_{k1} & \cdots & m_{kj}-1 & \cdots & m_{kk} \\ & \cdots & & & \\ m_{11} & & & & \end{vmatrix}. \tag{7b}$$

They obviously obey the canonical commutation relations

$$[\bar{a}_i^k, \bar{a}_j^l] = [a_i^k, a_j^l] = 0, \qquad [a_i^k, \bar{a}_j^l] = \delta_{ij}\,\delta_{kl} I; \tag{8}$$

the same is true for the operators

$$Q_j^k(\beta) = \bar{a}_j^k \cos\beta + a_j^k \sin\beta, \qquad P_j^k(\beta) = -\bar{a}_j^k \sin\beta + a_j^k \cos\beta. \tag{9}$$

Substituting now $Q_j^k(\beta)$, $P_j^k(\beta)$ into the formulae (5) for q_j^k, p_j^k, we obtain a representation of gl$(n+1,\mathbb{C})$ on V_{n+1} which depends on the parameters $\alpha_0, \alpha_1, \ldots, \alpha_n$ and β. In the following we shall deal mostly with the case $\beta = 0$. The representations of gl$(k+1,\mathbb{C})$ obtained in this way $(k = 1, 2, \ldots, n)$ will be denoted by $\rho^{(k+1)}$. We shall use also E_{ij}^{k+1} as a shorthand for $\rho^{(k+1)}(e_{ij})$, $e_{ij} \in$ gl$(k+1,\mathbb{C})$; in this notation the representation under consideration is given by the relations

$$E_{ij}^{n+1} = \bar{a}_i^{n+1} a_j^{n+1} + E_{ij}^n + \tfrac{1}{2}\delta_{ij} I, \tag{10a}$$

$$E_{n+1,i}^{n+1} = -a_i^{n+1}, \tag{10b}$$

$$E_{i,n+1}^{n+1} = \bar{a}_i^{n+1}\Big(\sum_{j=1}^n \bar{a}_j^{n+1} a_j^{n+1} + \frac{n+1}{2} - \mathrm{i}\alpha_n\Big) + \sum_{j=1}^n \bar{a}_j^{n+1} E_{ij}^n, \tag{10c}$$

$$E_{n+1,n+1}^{n+1} = -\sum_{j=1}^n \bar{a}_j^{n+1} a_j^{n+1} - \Big(\frac{n}{2} - \mathrm{i}\alpha_n\Big) I. \tag{10d}$$

Let us further express the corresponding representation of the subalgebra sl$(n+1,\mathbb{C})$ in terms of the basis (2). We obtain

$$(11a) = (10a), \qquad (11b) = (10b),$$

$$E_{i,n+1}^{n+1} = \bar{a}_i^{n+1}\Big(\sum_{j=1}^n \bar{a}_j^{n+1} a_j^{n+1} - \sum_{k=1}^{i-1} \bar{a}_k^i a_k^i + \sum_{r=i+1}^n \bar{a}_i^r a_i^r - \sum_{s=i}^n \Lambda_s\Big) + \sum_{i\neq j=1}^n \bar{a}_j^{n+1} E_{ij}^n, \tag{11c}$$

$$H_j^{n+1} \equiv E_{j+1,j+1}^{n+1} - E_{jj}^{n+1} = \sum_{r=j+2}^{n+1} (\bar{a}_{j+1}^r a_{j+1}^r - \bar{a}_j^r a_j^r) - 2\bar{a}_j^{j+1} a_j^{j+1}$$
$$- \sum_{s=1}^{j-1} (\bar{a}_s^{j+1} a_s^{j+1} - \bar{a}_s^j a_s^j) + \Lambda_j I, \tag{11d}$$

where

$$\Lambda_j = \mathrm{i}\alpha_j - \mathrm{i}\alpha_{j-1} - 1, \qquad j = 1, 2, \ldots, n. \tag{12}$$

These new parameters will be very important in the following. For any $\Lambda = (\Lambda_1, \Lambda_2, \ldots, \Lambda_n)$ the formulae (11) define a representation of $\mathrm{sl}(n+1, \mathbb{C})$; we call it the *maximal representation* and denote it by $\rho_\Lambda^{(n+1)}$ or simply ρ_Λ if there is no danger of misunderstanding.

3.2. Proposition.

The restriction $\tilde{\rho}_\Lambda^{(n+1)}$ of $\rho_\Lambda^{(n+1)}$ to the subspace $V_{n+1}^\Lambda \equiv \rho_\Lambda^{(n+1)}(\mathrm{UL})x_0^{n+1}$ of V_{n+1} is a representation of $\mathrm{L} = \mathrm{sl}(n+1, \mathbb{C})$ with the highest weight Λ and the vacuum x_0^{n+1} as its highest-weight vector.

Proof. The relations (11*d*) and (7*b*) imply $\rho_\Lambda(h_j)x_0^{n+1} = \Lambda_j x_0^{n+1}, j = 1, 2, \ldots, n$. Further

$$\rho_\Lambda(e_i) = \rho_\Lambda(e_{i+1,i}) = \sum_{k=i+2}^{n+1} \bar{a}_{i+1}^k a_i^k - a_i^{i+1},$$

due to (11*a*,*b*), so that $\rho_\Lambda(e_i)x_0^{n+1} = 0$, $i = 1, 2, \ldots, n$. The restriction $\tilde{\rho}_\Lambda^{(n+1)}$ is properly defined because $\rho_\Lambda(a)$ maps V_{n+1}^Λ into itself for any $a \in \mathrm{L}$. The condition (iii) of § 2.3 is fulfilled automatically for $\tilde{\rho}_\Lambda$.

Thus we have constructed for any $\Lambda = (\Lambda_1, \ldots, \Lambda_n)$ the highest-weight representation $\tilde{\rho}_\Lambda^{(n+1)}$ of $\mathrm{sl}(n+1, \mathbb{C})$. These representations are even irreducible, as we shall show a little later. However, they are not yet suitable for practical use, because we do not know the representation space V_{n+1}^Λ explicitly. In the next section we shall find conditions under which $V_{n+1}^\Lambda = V_{n+1}$, i.e. $\tilde{\rho}_\Lambda^{(n+1)} = \rho_\Lambda^{(n+1)}$; they turn out to be irreducibility conditions for $\rho_\Lambda^{(n+1)}$.

4. Irreducibility conditions for $\rho_\Lambda^{(n+1)}$

4.1

We shall use the following simple fact.

Proposition. Every non-trivial invariant subspace $V' \subset V_{n+1}$ of $\rho_\Lambda^{(n+1)}$ contains the vacuum vector x_0^{n+1}.

Proof. Since V' is assumed to be non-trivial it contains at least one non-zero vector $x \in V_{n+1}$. We can write

$$x = \sum_m \alpha_m \begin{vmatrix} m_{n1} & \ldots & m_{nn} \\ & \ldots & \\ m_{11} & & \end{vmatrix},$$

$$m = (m_{n1}, \ldots, m_{nn}, m_{n-1,1}, \ldots, m_{n-1,n-1}, m_{n-2,1}, \ldots, m_{11}).$$

Let $\bar{m} = (\bar{m}_{n1}, \ldots, \bar{m}_{11})$ be a 'highest degree' of this sum, understood in the following sense:

$$\begin{aligned} \bar{m}_{n1} &= \max\{m_{n1} : \alpha_m \neq 0\}, \\ \bar{m}_{n2} &= \max\{m_{n2} : \alpha_{\bar{m}_{n1} m_{n2} \ldots m_{11}} \neq 0\}, \\ &\ldots \\ \bar{m}_{11} &= \max\{m_{11} : \alpha_{\bar{m}_{n1} \ldots \bar{m}_{22} m_{11}} \neq 0\}. \end{aligned} \qquad (*)$$

The relation (11*a*) implies

$$E_{ij}^{n+1} = \sum_{k=i+1}^{n+1} \bar{a}_i^k a_j^k - a_j^i, \qquad i>j. \tag{11e}$$

Since V' is assumed to be an invariant subspace of ρ_Λ, the vector $E_{ij}^{n+1}y$ belongs to V' for any $y \in V'$. Consequently, the vector

$$\bar{x} = (E_{21}^{n+1})^{\bar{m}_{11}}(E_{32}^{n+1})^{\bar{m}_{22}}(E_{31}^{n+1})^{\bar{m}_{21}}(E_{43}^{n+1})^{\bar{m}_{33}} \dots$$
$$\dots (E_{n1}^{n+1})^{\bar{m}_{n-1,1}}(E_{n+1,m}^{n+1})^{\bar{m}_{nn}} \dots (E_{m+1,1}^{n+1})^{\bar{m}_{n1}}x$$

belongs to V'. The chosen order ensures that the sums from (11*e*) do not contribute. Further, (*) together with (7*b*) imply $\bar{x} = c\alpha_{\bar{m}}x_0^{n+1}$, where c is some non-zero number (a product of powers of -1 and square roots of positive numbers); therefore $x_0^{n+1} \in V'$.

4.2. Corollary.

The representation $\tilde{\rho}_\Lambda^{(n+1)}$ from proposition 3.2 is irreducible for any Λ.

Proof. Any non-trivial invariant subspace $V' \subset V_{n+1}^\Lambda$ of $\tilde{\rho}_\Lambda$ is at the same time invariant under ρ_Λ; thus it contains the vacuum vector x_0^{n+1}. It further implies $V^1 \equiv \rho_\Lambda(L)x_0^{n+1} \subset V'$, $V^2 \equiv \rho_\Lambda(L)V^1 \subset V'$ etc. We obtain therefore $\rho_\Lambda(\mathrm{UL})x_0^{n+1} = V_{n+1}^\Lambda \subset V'$ so that $V' = V_{n+1}^\Lambda$.

4.3

Let us turn now to the problem mentioned at the end of the last section. We shall prove the following assertion.

Theorem. Let the conditions

$$(\Lambda_j + \Lambda_{j-1} + \dots + \Lambda_k + j - k) \notin N_0 \tag{13}$$

be satisfied for any pair of integers j, k, $1 \leq k \leq j = 1, 2, \dots, n$. Then the maximal representation $\rho_\Lambda^{(n+1)}$ of sl$(n+1, \mathbb{C})$ defined by the formulae (11) is irreducible; it has the highest weight $\Lambda = (\Lambda_1, \dots, \Lambda_n)$ and the highest-weight vector x_0^{n+1}.

Proof. In view of proposition 3.2 and corollary 4.2 it is sufficient to verify that under the stated assumptions $V_{n+1}^\Lambda = V_{n+1}$ holds. We shall do it in several steps.

4.4

We denote first by x_s^{n+1} the basis vectors (6) with all the indices equal to zero with the exception of $m_{n1}: m_{kj} = s\,\delta_{kn}\,\delta_{j1}$; in particular, x_0^{n+1} is the vacuum vector as before. The following assertion holds.

Lemma. If the conditions (13) are valid for $1 \leq k \leq j = n$, then the subspace V_{n+1}^Λ contains the vectors x_s^{n+1} for all $s \in N_0$.

Proof. Let us introduce the following finite sequences of operators (for convenience written as columns):

$$R_1 = I, \qquad R_{k+1} = \begin{vmatrix} R_1 E^{n+1}_{1,k+1} \\ R_2 E^{n+1}_{2,k+1} \\ \vdots \\ R_k E^{n+1}_{k,k+1} \end{vmatrix}, \qquad k = 1, 2, \ldots, n. \tag{*}$$

Let us further take an arbitrary $s \in N_0$ and denote

$$y_n \equiv R_{n+1} x^{n+1}_s = \begin{vmatrix} R_1 E^{n+1}_{1,n+1} x^{n+1}_s \\ R_2 E^{n+1}_{2,n+1} x^{n+1}_s \\ \vdots \\ R_n E^{n+1}_{n,n+1} x^{n+1}_s \end{vmatrix}.$$

This column has 2^{n-1} rows. We divide it into two parts with 2^{n-2} rows:

$$y^1_n = \begin{vmatrix} R_1 E^{n+1}_{1,n+1} x^{n+1}_s \\ \vdots \\ R_{n-1} E^{n+1}_{n-1,n+1} x^{n+1}_s \end{vmatrix}, \qquad y^2_n = |R_n E^{n+1}_{n,n+1} x^{n+1}_s|,$$

and introduce

$$y_{n-1} = (s - \Lambda_n) y^1_n - y^2_n.$$

Further we divide the column y_{n-1} into two parts: y^1_{n-1} (consisting of the first 2^{n-3} rows) and y^2_{n-2}. Then we define $y_{n-2} = (s - \Lambda_n - \Lambda_{n-1} - 1) y^1_{n-1} - y^2_{n-1}$. Continuing this procedure, we put

$$y_{n-k} = (s - \Lambda_n - \Lambda_{n-1} - \ldots - \Lambda_{n-k+1} - k + 1) y^1_{n-k+1} - y^2_{n-k+1}$$

for all $1 \leqslant k \leqslant n-1$. Finally we obtain a one-row column, i.e. a vector y_1. We shall prove that the relation

$$y_1 = (s+1)^{1/2} \prod_{j=1}^{n} (s - \Lambda_n - \Lambda_{n-1} - \ldots - \Lambda_j - n + j) x^{n+1}_{s+1} \tag{**}$$

holds. For this purpose we shall use the relations (11c) together with (7a,b). The latter imply

$$\bar{a}^{k+1}_j a^{k+1}_j x^{n+1}_s = s\, \delta_{kn}\, \delta_{j1} x^{n+1}_s, \qquad a^{n+1}_j x^{n+1}_s = \sqrt{s}\, \delta_{kn}\, \delta_{j1} x^{n+1}_{s-1}, \tag{***}$$

so that we obtain for y_n the following expression:

$$\begin{vmatrix} R_1 \bar{a}^{n+1}_1 (s - \Lambda_n - \ldots - \Lambda_1) x^{n+1}_s & & + R_1 \sum_{j=2}^{n} \bar{a}^{n+1}_j E^n_{1j} x^{n+1}_s \\ R_2 \bar{a}^{n+1}_2 (s - \Lambda_n - \ldots - \Lambda_2) x^{n+1}_s & + R_2 \bar{a}^{n+1}_1 E^n_{21} x^{n+1}_s & + R_2 \sum_{j=3}^{n} \bar{a}^{n+1}_j E^n_{2j} x^{n+1}_s \\ \vdots & & \\ R_{n-1} \bar{a}^{n+1}_{n-1} (s - \Lambda_n - \Lambda_{n-1}) x^{n+1}_s & + R_{n-1} \sum_{j=1}^{n-2} \bar{a}^{n+1}_j E^n_{n-1,j} x^{n+1}_s & + R_{n-1} \bar{a}^{n+1}_n E^n_{n-1,n} x^{n+1}_s \\ R_n \bar{a}^{n+1}_n (s - \Lambda_n) x^{n+1}_s & + R_n \sum_{j=1}^{n-1} \bar{a}^{n+1}_j E^n_{nj} x^{n+1}_s & \end{vmatrix}.$$

Here the terms containing $E^n_{ij}x^{n+1}_s$, $i>j$, are equal to zero because of (***) and the relation

$$E^n_{ij} = \sum_{k=i+1}^{n} \bar{a}^k_i a^k_j - a^i_j, \qquad i>j,$$

which is obtained in the same way as (11e). Now we substitute for R_n from (*); using further the relations (11a), (8) and (***) we obtain

$$\left|\begin{array}{ll} R_1\bar{a}^{n+1}_1(s-\Lambda_n-\ldots-\Lambda_1)x^{n+1}_s + R_1\sum\limits_{j=2}^{n-1}\bar{a}^{n+1}_j E^n_{1j}x^{n+1}_s & + R_1\bar{a}^{n+1}_n E^n_{1n}x^{n+1}_s \\ R_2\bar{a}^{n+1}_2(s-\Lambda_n-\ldots-\Lambda_2)x^{n+1}_s + R_2\sum\limits_{j=3}^{n-1}\bar{a}^{n+1}_j E^n_{2j}x^{n+1}_s & + R_2\bar{a}^{n+1}_n E^n_{2n}x^{n+1}_s \\ \vdots & \\ R_{n-1}\bar{a}^{n+1}_{n-1}(s-\Lambda_n-\Lambda_{n-1})x^{n+1}_s & + R_{n-1}\bar{a}^{n+1}_n E^n_{n-1,n}x^{n+1}_s \\ R_1\bar{a}^{n+1}_1(s-\Lambda_n)x^{n+1}_s & +(s-\Lambda_n)R_1\bar{a}^{n+1}_n E^n_{1n}x^{n+1}_s \\ \vdots & \\ R_{n-1}\bar{a}^{n+1}_{n-1}(s-\Lambda_n)x^{n+1}_s & +(s-\Lambda_n)R_{n-1}\bar{a}^{n+1}_n E^n_{n-1,n}x^{n+1}_s \end{array}\right| .$$

Subtracting the lower half of this column from the upper one multiplied by $(s-\Lambda_n)$, we obtain the following expression for y_{n-1}:

$$\left|\begin{array}{l} R_1\bar{a}^{n+1}_1(s-\Lambda_n)(s-\Lambda_n-\ldots-\Lambda_1-1)x^{n+1}_s + R_1(s-\Lambda_n)\sum\limits_{j=2}^{n-1}\bar{a}^{n+1}_j E^n_{1j}x^{n+1}_s \\ R_2\bar{a}^{n+1}_2(s-\Lambda_n)(s-\Lambda_n-\ldots-\Lambda_2-1)x^{n+1}_s + R_2(s-\Lambda_n)\sum\limits_{j=3}^{n-1}\bar{a}^{n+1}_j E^n_{2j}x^{n+1}_s \\ \vdots \\ R_{n-1}\bar{a}^{n+1}_{n-1}(s-\Lambda_n)(s-\Lambda_n-\Lambda_{n-1}-1)x^{n+1}_s \end{array}\right| .$$

In the next step we substitute for R_{n-1} from (*); then we use again the relations (11a), (8) and (***) and subtract the lower half from the upper one multiplied by $(s-\Lambda_n-\Lambda_{n-1})$:

$$y_{n-2} = (s-\Lambda_n)(s-\Lambda_n-\Lambda_{n-1}-1)$$
$$\times\left|\begin{array}{l} R_1\bar{a}^{n+1}_1(s-\Lambda_n-\ldots-\Lambda_1-2)x^{n+1}_s + R_1\sum\limits_{j=2}^{n-2}\bar{a}^{n+1}_j E^n_{1j}x^{n+1}_s \\ R_2\bar{a}^{n+1}_2(s-\Lambda_n-\ldots-\Lambda_2-2)x^{n+1}_s + R_2\sum\limits_{j=3}^{n-2}\bar{a}^{n+1}_j E^n_{2j}x^{n+1}_s \\ \vdots \\ R_{n-2}\bar{a}^{n+1}_{n-2}(s-\Lambda_n-\Lambda_{n-1}-\Lambda_{n-2}-2)x^{n+1}_s \end{array}\right| .$$

Repeating this procedure, we obtain finally

$$y_1 = \prod_{j=1}^{n}(s-\Lambda_n-\Lambda_{n-1}-\ldots-\Lambda_j-n+j)\bar{a}^{n+1}_1 x^{n+1}_s,$$

i.e. the formula (**). The presented construction shows that there exists an element $p\in$ UL such that

$$\rho_\Lambda(p)x^{n+1}_s = (s+1)^{1/2}\prod_{j=1}^{n}(s-\Lambda_n-\ldots-\Lambda_j-n+j)x^{n+1}_{s+1}.$$

This vector is non-zero due to the assumption, thus if x_s^{n+1} belongs to the subspace $V_{n+1}^{\Lambda}=\rho_\Lambda(\mathrm{UL})x_0^{n+1}$ then the same is true for x_{s+1}^{n+1}. Since x_0^{n+1} is contained in V_{n+1}^{Λ}, the proof is completed by induction.

4.5

Now we can continue the proof of theorem 4.3. Since $V_{n+1}^{\Lambda}\subset V_{n+1}$, we have to prove the opposite inclusion; it is clearly sufficient to verify that all the basis vectors (6) of V_{n+1} are contained in V_{n+1}^{Λ}. We decompose V_{n+1} in the following way: let D_{n+1} be the set of all symbols $l=(l_1, l_2, \ldots, l_n)$, $l_i\in N_0$ and L_{n+1} be the complex vector space spanned by D_{n+1}. Then we can write

$$V_{n+1}=L_{n+1}\otimes V_n, \qquad x_0^{n+1}=l_0\otimes x_0^n, \tag{14}$$

where

$$(l_1,\ldots,l_n)\otimes\begin{vmatrix} m_{n-1,1} & \ldots & m_{n-1,n-1} \\ & \ldots & \\ m_{11} & & \end{vmatrix}\equiv\begin{vmatrix} l_1 & \ldots & l_{n-1} & l_n \\ m_{n-1,1} & \ldots & m_{n-1,n-1} & \\ & \ldots & & \\ m_{11} & & & \end{vmatrix}$$

and $l_0\equiv(0,\ldots,0)$. We shall prove first

$$L_{n+1}\otimes x_0^n\subset V_{n+1}^{\Lambda}. \tag{15}$$

Let us take $l=(l_1,\ldots,l_k,0,\ldots,0)$. $1\leqslant k\leqslant n-1$. Using (11$e$) we obtain

$$(E_{k+1,k}^{n+1})^r(l\otimes x_0^n)=\binom{l_k}{r}^{1/2} r!((l_1,\ldots,l_{k-1},l_k-r,r,0,\ldots,0)\otimes x_0^n).$$

According to lemma 4.4 the vectors $x_s^{n+1}=(s,0,\ldots,0)\otimes x_0^n$ belong to V_{n+1}^{Λ}, so acting on them by powers of $E_{k+1,k}^{n+1}$ we stay within V_{n+1}^{Λ}. Starting with s large enough, we can obtain in this way every basis vector of $L_{n+1}\otimes x_0^n$; thus the relation (15) holds.

4.6

Further we shall show that V_{n+1}^{Λ} contains $L_{n+1}\otimes x_r^n$ for any $r\in N_0$. We know that this is true for $r=0$; let us assume the same for $r=1,2,\ldots,s$. The proof is analogous to that of lemma 4.4: we start with an arbitrary $\tilde{x}_s=l\otimes x_s^n$ and denote $\tilde{y}_{n-1}=R_n\tilde{x}_s$. Then we divide this column into two parts $\tilde{y}_{n-1}^1$, $\tilde{y}_{n-1}^2$ and define $\tilde{y}_{n-2}=(s-\Lambda_{n-1})\tilde{y}_{n-1}^1-\tilde{y}_{n-1}^2$. Continuing this procedure with

$$\tilde{y}_{n-k}=(s-\Lambda_{n-1}-\ldots-\Lambda_{n-k+1}-k+2)\tilde{y}_{n-k+1}^1-\tilde{y}_{n-k+1}^2$$

we arrive finally at the vector $\tilde{y}_1$. According to the construction this vector belongs to V_{n+1}^{Λ}; we shall show it to be of the form

$$\tilde{y}_1=(s+1)^{1/2}\prod_{j=1}^{n-1}(s-\Lambda_{n-1}-\ldots-\Lambda_j-n+j-1)(l\otimes x_{s+1}^n)+\tilde{x}_s' \tag{*}$$

where $\tilde{x}'_s$ is some vector from $L_{n+1} \otimes x^n_s$. We write $\tilde{y}_{n-1}$ using (11*a*). Then we express E^n_{ij} with the help of relations (11*c*) and (7), obtaining thus for $\tilde{y}_{n-1}$

$$\left| \begin{matrix} R_1 \bar{a}^n_1 (s-\Lambda_{n-1}-\ldots-\Lambda_1)\tilde{x}_s & + R_1 \sum_{j=2}^{n-1} \bar{a}^n_j E^{n-1}_{1j} \tilde{x}_s + \bar{a}^{n+1}_1 a^{n+1}_n \tilde{x}_s \\ R_2 \bar{a}^n_2 (s-\Lambda_{n-1}-\ldots-\Lambda_2)\tilde{x}_s + R_2 \bar{a}^n_1 E^{n-1}_{21} \tilde{x}_s & + R_2 \sum_{j=3}^{n-1} \bar{a}^n_j E^{n-1}_{2j} \tilde{x}_s + \bar{a}^{n+1}_2 a^{n+1}_n \tilde{x}_s \\ \vdots & \\ R_{n-1} \bar{a}^n_{n-1} (s-\Lambda_{n-1})\tilde{x}_s + R_{n-1} \sum_{j=1}^{n-2} \bar{a}^n_j E^{n-1}_{n-1,j} \tilde{x}_s & + \bar{a}^{n+1}_{n-1} a^{n+1}_n \tilde{x}_s \end{matrix} \right| .$$

Now we can proceed further in the same way as in the proof of lemma 4.4. The added vectors $\bar{a}^{n+1}_j a^{n+1}_n \tilde{x}_s$ belong to $L_{n+1} \otimes x^n_s$, thus the same holds for any linear combination of them. Finally we obtain

$$\tilde{y}_1 = \prod_{j=1}^{n-1} (s-\Lambda_{n-1}-\ldots-\Lambda_j-n+j-1)\bar{a}^n_1 \tilde{x}_s + \tilde{x}'_s,$$

$$\tilde{x}'_s = \prod_{k=2}^{n-1} (s-\Lambda_{n-1}-\ldots-\Lambda_{n-k+1}-k+2)\bar{a}^{n+1}_1 a^{n+1}_1 \tilde{x}_s$$

$$-\sum_{j=2}^{n-1} \prod_{k=2}^{n-j} (s-\Lambda_{n-1}-\ldots-\Lambda_{n-k+1}-k+2)\bar{a}^{n+1}_j a^{n+1}_n \tilde{x}_s;$$

this proves the relation (*). The induction assumption implies $\tilde{x}'_s \in V^\Lambda_{n+1}$; then the vector $\tilde{y}_1 - \tilde{x}'_s$ also belongs to V^Λ_{n+1}. The conditions (13) are assumed to be valid for $j = 1, 2, \ldots, n$, especially for $j = n-1$, and thus $\tilde{y}_1 - \tilde{x}'_s$ is a non-zero multiple of $\tilde{x}_{s+1}$, which belongs therefore to V^Λ_{n+1}.

4.7

Further, we decompose V_{n+1} into the tensor product $V_{n+1} = L_{n+1} \otimes L_n \otimes V_{n-1}$ in analogy with (14). Let us take some $k = 1, 2, \ldots, n-2$ and natural r and assume the vectors $x(m, 1(k,s)) \equiv m \otimes l(k,s) \otimes x^{n-1}_0$ to belong to V^Λ_{n+1} for any $m \in L_{n+1}$, $l(k,s) = (l_1, \ldots, l_k, s, 0, \ldots, 0)$, l_i arbitrary elements of N_0 and $s = 0, 1, \ldots, r-1$. The relations (7) and (11*e*) imply

$$(E^{n+1}_{k+1,k})^r x(m, l(k,0))$$
$$= \sum_{j=0}^{r} r! \binom{m_k}{r-j}^{1/2} \binom{m_{k+1}+r-j}{r-j}^{1/2} \binom{l_k}{j}^{1/2}$$
$$\times (m_1, \ldots, m_k - r, m_{k+1}+r, \ldots, m_n) \otimes (l_1, \ldots, l_k - j, j, 0, \ldots, 0) \otimes x^{n-1}_0.$$

Thus by induction V^Λ_{n+1} contains the vectors $x(m, l(k,r))$ for all $r \in N_0$, i.e. if V^Λ_{n+1} contains the vectors $x(m, l(k,0))$ with arbitrary $l(k,0)$ and $m \in L_{n+1}$, then the same is true for $x(m, l(k+1, 0))$. According to § 4.6 the vectors $x(m, l(1,0))$ belong to V^Λ_{n+1}, so that applying once more the induction argument we obtain $x(m, l(n-1, 0)) = x(m, l) = m \otimes l \otimes x^{n-1}_0 \in V^\Lambda_{n+1}$ for any $m \in L_{n+1}$, $l \in L_n$.

4.8

Now one has to repeat the considerations of §§ 4.6 and 4.7 in order to 'fill up' the third row. Continuing this procedure, we arrive at the relation

$$V_{n+1} = \bigotimes_{k=1}^{n} L_{k+1} \subset V_{n+1}^{\Lambda}$$

which represents the desired result.

5. Discussion

5.1

Let us assume all the components of the highest weight to be real. Then coefficients of all polynomials used in the performed proofs are also real. It means that in this case the results obtained in the previous sections for $\mathrm{sl}(n+1, \mathbb{C})$ apply to the real form $\mathrm{sl}(n+1, \mathbb{C})$ as well; this question is left open for complex weights. The problem naturally arises under which conditions the representations of $\mathrm{sl}(n+1, \mathbb{R})$ and other real forms of A_n are skew-symmetric. The answer depends on a choice of the inner product in V_{n+1} or, more generally, on a representation of the canonical pairs substituted into (5). There are some results which can be used here, for example the following statement which is concluded easily from Havlíček and Lassner (1975): if the parameters $\alpha_0, \ldots, \alpha_n$ in (5) are real and q_j^{k+1}, p_j^{k+1} are represented by symmetric and skew-symmetric operators (on a common dense invariant domain), respectively, then the resulting representation of $\mathrm{sl}(n+1, \mathbb{R})$ is skew-symmetric. A detailed discussion of this problem is left to the subsequent paper.

5.2

The finite-dimensional irreducible representations of $\mathrm{sl}(n+1, \mathbb{C})$ may be described completely in the framework of Gel'fand–Zetlin patterns. There exists a generalisation of this method (Barut and Raczka 1977, § 11.8) which makes it possible to construct also some infinite-dimensional highest-weight representations. In the case of $\mathrm{sl}(3, \mathbb{C})$, for example, one has to replace the Gel'fand–Zetlin patterns

$$\begin{pmatrix} m_{13} & & m_{23} & & m_{33} \\ & m_{12} & & m_{22} & \\ & & m_{11} & & \end{pmatrix} \quad \text{by} \quad \begin{pmatrix} & m_{13} & & m_{23} & & m_{33} \\ m_{12} & & m_{22} & & & \\ & m_{11} & & & & \end{pmatrix}$$

with $m_{12} \geqslant m_{13}+1$, $m_{13} \geqslant m_{22} \geqslant m_{23}$, $m_{12} \geqslant m_{11} \geqslant m_{22}$. Action of the standard Gel'fand–Zetlin formulae on these patterns defines an infinite-dimensional highest-weight representation of $\mathrm{sl}(3, \mathbb{C})$ (determined by m_{13}, m_{23}, m_{33}). However, one can obtain in this way only representations with (possibly negative) integer components of the highest weight; they correspond only to a small subset of the representations which we have studied here.

5.3

We have to compare our results first of all with the elementary representations introduced in § 2.6, because the latter are defined also for each weight Λ on H.

(a) Every d_Λ is the highest-weight representation; for our maximal representations this is true if the conditions (13) are satisfied, otherwise we obtain the highest-weight representation $\tilde{\rho}_\Lambda$ by restriction of ρ_Λ to the subspace V^Λ_{n+1}. On the other hand, the highest-weight representations which we obtain are always irreducible. This difference is due to different incomplete reducibility of d_Λ and ρ_Λ: symbolically

$$d_\Lambda = \begin{pmatrix} \tilde{d}_\Lambda & 0 \\ * & d'_\Lambda \end{pmatrix}, \qquad \rho_1 = \begin{pmatrix} \tilde{\rho}_\Lambda & * \\ 0 & \rho'_\Lambda \end{pmatrix},$$

and $\tilde{d}_\Lambda$ and $\tilde{\rho}_\Lambda$ are the irreducible components of d_Λ and ρ_Λ, respectively, and the asterisk stands for non-zero blocks.

(b) Action of the operators $\rho_\Lambda(h_i)$, $\rho_\Lambda(e_{ij})$ on an arbitrary vector of V_{n+1} is obtained from the formulae (7) and (11). In particular, they allow us to calculate easily matrix elements of the generators. This is not true for the elementary representations for which the choice of a basis in the representation space is itself complicated. According to our opinion, this fact represents the main advantage of the maximal representations. We pay, of course, a price for it: the formulae (4) defining elementary representations are common for all the complex semi-simple Lie algebras, while ours refer to the algebras A_n only. There exists, however, a hope of performing an analogous construction for the remaining classical semi-simple Lie algebras.

(c) In the subsequent paper mentioned in § 1.3 we shall illustrate the irreducibility conditions on the example of sl$(3, \mathbb{C})$. We shall prove that the conditions (13) are in this case necessary and sufficient for irreducibility of the maximal as well as the elementary representations. Further, we shall show that starting from the canonical realisations (5) one can construct irreducible highest-weight representations also for some of the weights such that the corresponding elementary representations are reducible.

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